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Eigenvalue Spectrum of a Dirac Particle in Static and Spherical Complex Potential

Khaled Saaidi[1,2]

It has been observed that a quantum theory need not be Hermitian to have a real spectrum. We study the non-Hermitian relativistic quantum theories for many complex potentials, and obtain the real relativistic energy eigenvalues and corresponding eigenfunctions of a Dirac-charged particle in complex statically and spherically symmetric potentials. Complex Dirac–Eckart, complex Dirac–Rosen–Morse II, complex Dirac–Scarf and complex Dirac–Poschl–Teller potential are investigated.

KEY WORDS: eigenvalue spectrum; Dirac particle; complex potential.

1. INTRODUCTION

The first study carried out in the non-Hermitian quantum theory dates back to an old paper by Caliceti *et al.* (1980). The imaginary cubic oscillator problem in the context of perturbation theory has been studied in Caliceti *et al.* (1980). The energy spectrum of that model is real and discrete. It shows that one may construct many new Hamiltonians that have real spectrum, although their Hamiltonians are not Hermitian. The key idea of the new formalism (non-Hermitian quantum theory) lies in the empirical observation that the existence of the real spectrum needs not to necessarily be attributed to the Hermiticity of the Hamiltonian. Such non-Hermitian formalism, for the context of Schrödinger Hamiltonian has been studied for several models in (Caliceti *et al.*, 1980; Bender *et al.*, 1997, 1999, 2001, 2002; Saaidi, Trinh and Delabaerer, 2000; Delabaerer and Pham, 1998; Dorey *et al.*, 2001; Mezincescu, 2000; Benda and Wang, 2001; Znogil and Lavai, 1999, 2000; Bender and Weniger, 2001; Mostafazadeh, 2002; Bagchi and Quesne, 2000, 2002; Bogchi *et al.*, 2001; Ahmed, 2001; Znojil, 1999; Mostafazadeh, 2003; Weigert; Bidenharn, 1962). The analysis of the related purely real spectra of energies has been performed with different techniques. For example, resummations of divergent perturbation

[1] Department of Science, University of Kurdistan, Pasdaran Ave., Sanandaj, Iran; e-mail: K. Saaidi @ipm.ir, ksaaidi@uok.ac.ir.
[2] Institute for Studies in Theoretical Physics and Mathematics, P.O. Box, Tehran 19395-5531, Iran.

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series (Caliceti *et al.*, 1980), delta expansions (Bender, Boettcher, and Meisinger, 1999) WKB method (Delabaere and Pham, 1998) functional analysis (Mezincescu, 2000), pseudo-Hermitian methods (Mostafazadeh, 2002, 2003; Ahmed, 2001), and complex Lee algebra (Bagchi and Quesne, 2000, 2002; Bagchi, Mallik and Quesne, 2001). Some explicit studies of the Hermitian and non-Hermitian Hamiltonians have also been performed in the context of Dirac Hamiltonian. For example, the solution of ordinary (Hermitian) Dirac equation for Coulomb potential, including its relativistic bound state spectrum and wave function, was investigated in (Znojil, 1999; Mostafazadeh, 2003). Also, by adding off-diagonal real linear radial term to the ordinary Dirac operator, the relativistic Dirac equation with oscillator potential has been introduced (Weigert, Bidenharn, 1962) then the energy spectrum of corresponding eigenfunctions, has been obtained. The ordinary (Hermitian) Dirac equation for a charged particle in static electromagnetic field is studied for Morse potential (Alhaidari, 2002). The non-Hermitian formalism, for the context of Dirac Hamiltonian, has been studied for three dimensional complex Dirac–Morse and complex Dirac–Coulomb potentials in (Mastafa, 2003; Saaidi, Submitted).

In this paper, we consider the non-Hermitian Dirac Hamiltonian for several complex potentials. We study a charged particle in statically and spherically symmetric four vector complex potentials. By applying a unitary transformation to Dirac equation, we obtain the second order Schrödinger-like equation, therefore; comparison with well-known non-relativistic problems is transparent. Using correspondence between parameters of the two problems (the Schrödinger equation and the Schrödinger-like equation which is obtained after applying the unitary transformation to Dirac equation for a potential), we obtain the bound states spectrum and corresponding eigenfunctions.

The scheme of this article is as follows: in Section 2, we study the non-Hermitian version of Dirac equation for a charged particle with static and spherically symmetric potential, and by applying a unitary transformation, we obtain the proper gauge fixing condition and Schrödinger like differential equation. In Section 3, we discuss the Dirac equation for complex Dirac–Eckart potential and obtain the real energy spectrum and corresponding eigenfunctions. In Sec. 4, we consider the Dirac equation for a complex Dirac–Rosen–Morse II potential, then, we obtain the real eigenvalue and its wave function. In Sec. 5, and 6, we study the complex Dirac–Scarf and Dirac–Poschl–Teller potentials, respectively. We obtain the relativistic energy spectrum and corresponding wave function for the upper component of spinors.

2. PRELIMINARIES

The Hamiltonian of a Dirac particle for a complex electromagnetic field is ($c = \hbar = 1$)

$$H = \hat{\alpha}(\hat{p} - e\hat{A}) + \hat{\beta}m + eV \tag{1}$$

where the Dirac matrices $\hat{\alpha}$, $\hat{\beta}$ have their usual meaning, and setting A_0 equal to V. In (1), $\hat{A}$ and V are the vector and scalar complex field, respectively, where, $\hat{A}^* \neq \hat{A}$ and $V^* \neq V$. Then the Dirac Hamiltonian (1) is not Hermitian. It is well known that the local gauge symmetry in quantum electrodynamic implies an invariance under the transformation as

$$(V, \hat{A}) \to (V', \hat{A}') = \left(V + \frac{\partial \Lambda}{\partial t}, \hat{A} + \vec{\nabla}\Lambda\right) \tag{2}$$

Here $\Lambda(t, \vec{r})$ is a complex scalar field. Suppose that the charge distribution is static with spherical symmetry, so the gauge invariance implies that $V' = V$ and $\hat{A}' = \hat{r} A(r)$, where $\hat{r}$ is the radial unit vectror (Alhaidari, 2002). One can denoted the correspondence wave function of (1) as

$$\Psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix} \tag{3}$$

In this case one can obtain

$$\begin{aligned} (m + eV - E_{\mathrm{r}})\Phi &= i^{\lceil}\hat{\sigma} \cdot \vec{\nabla} - e(\hat{\sigma} \cdot \hat{r})A(r)^{\rceil}\chi \\ (eV - m - E_{\mathrm{r}})\chi &= i^{\lceil}\hat{\sigma} \cdot \vec{\nabla} + e(\hat{\sigma} \cdot \hat{r})A(r)^{\rceil}\Phi \end{aligned} \tag{4}$$

Here $\hat{\sigma}$'s are the three Pauli spin matrices, E_{r} is relativistic energy, and we replaced $ie\,\hat{\sigma} \cdot \hat{A}(-ie\hat{\sigma} \cdot \hat{A})$ in first (second) equation of (4) instead of $e\hat{\sigma} \cdot \hat{A}$, respectively. Note that, because of the spherical symmetry of the complex field, the angular-momentum operator $\hat{J}$, and the parity operator, $\hat{P}$, commute with the Hamiltonian and the two spinors Φ and χ have also opposite parity. So the correspondence wave functions are denoted by

$$\begin{aligned} \Phi &= ig(r)\Omega_{\kappa,\mu}(\vartheta, \varphi) \\ \chi &= f(r)\sigma_{\mathrm{r}}\Omega_{-\kappa,\mu}(\vartheta, \varphi) \end{aligned} \tag{5}$$

It is seen that

$$(\hat{\sigma} \cdot \vec{\nabla})ig(r)\Omega_{\kappa,\mu}(\vartheta, \varphi) = i\sigma_{\mathrm{r}}\Omega_{\kappa,\mu}\left(\partial_{\mathrm{r}} + \frac{1}{r} + \frac{\kappa}{r}\right)g(r) \tag{6}$$

$$(\hat{\sigma} \cdot \vec{\nabla})(f(r)\sigma_{\mathrm{r}}\Omega_{-\kappa,\mu}(\vartheta, \varphi)) = \sigma_{\mathrm{r}}\Omega_{-\kappa,\mu}\left(\partial_{\mathrm{r}} + \frac{1}{r} + \frac{\kappa}{r}\right)f(r) \tag{7}$$

where $\hat{\kappa}$ is the spin orbit coupling operator as

$$\hat{\kappa} = \hat{\sigma} \cdot \hat{L} + \hbar I \tag{8}$$

and we have used from

$$\hat{\kappa}\Omega_{\mp\kappa,\mu}(\vartheta, \varphi) = \pm\kappa\hbar\Omega_{\mp\kappa,\mu}(\vartheta, \varphi) \tag{9}$$

in which

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right) & \text{for } j = l+\frac{1}{2} \\ l = \left(j+\frac{1}{2}\right) & \text{for } j = l-\frac{1}{2} \end{cases} \tag{10}$$

Therefore, by defining $u_1 = g(r)/r, u_2 = f(r)/r$, we obtain the following two component radial Dirac equation [32]

$$(m + eV - E_{\rm r})u_1(r) = \left(\partial_{\rm r} - \frac{k}{r} - eA(r)\right) u_2(r)$$

$$(eV - m - E_{\rm r})u_2(r) = -\left(\partial_{\rm r} + \frac{k}{r} + eA(r)\right) u_1 \tag{11}$$

Note that, $A(r)$ is a guage field, which has a symmetry such as (2), therefore, it must be fixed. It is seen that fixing this gauge degree of freedom by $\vec{\nabla}\cdot\vec{A} \equiv \frac{\partial A}{\partial r} = 0$ is not a suitable choice. Remark that in this paper instead of solving Dirac equation we want to solve the second-order differential equation, which is obtained by eliminating one component of Eq. (11). However, for the cases which $eV \neq 0$, the second order differential equation is not Schrödinger like, and therefore, one can obtain the proper gauge fixing by applying the global unitary transformation on two components u_1 and u_2 such as

$$U = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \tag{12}$$

where $a, b \in \Re$, and $a^2 + b^2 = 1$. By applying (12) to the upper component, $\phi^{\rm u}$, and lower component, $\phi^{\rm l}$ of spinor and institute them in (11), we have

$$(m - E_{\rm r}C)\phi^{\rm u} + \left[\frac{i(S^2 - C^2)}{S}eV - iSE_{\rm r} - \partial_{\rm r}\right]\phi^{\rm l} = 0$$

$$\left[\frac{i(S^2 - C^2)}{S}eV - iSE_{\rm r} + \partial_{\rm r}\right]\phi^{\rm u} - (\dot{m} + E_{\rm r}C)\phi^{\rm l} = 0 \tag{13}$$

where, S = 2ab, $C = a^2 - b^2$ and we have used from a gauge-fixing condition as

$$eV = \frac{iS}{C}\left(eA + \frac{k}{r}\right) \tag{14}$$

However, we eliminate the $\phi^{\rm l}$ component in (13), and obtain the radial differential equation for $\phi^{\rm u}$ as

$$-\frac{d^2\phi^{\rm u}}{dr^2} + V_{\rm eff}\phi^{\rm u} + \left(m^2 - E_{\rm r}^2\right)\phi^{\rm u} = 0 \tag{15}$$

where

$$V_{\rm eff} = -\frac{(S^2 - C^2)^2}{S^2}(eV)^2 + 2E_{\rm r}(S^2 - C^2)(eV) - i\frac{(S^2 - C^2)}{S}\frac{d(eV)}{dr} \tag{16}$$

Furthermore, from (11), it is easily seen that for the cases which $eV = 0$ the unitary transformation is not necessary. So, we can rewrite u_1 and u_2 as

$$u_1 = \phi^{\mathrm{u}}$$
$$u_2 = \phi^{\mathrm{l}}$$

then, by eliminating ϕ^{l}, one can obtain the Schrödinger-like differential equation for radial upper component, ϕ^{u} as

$$-\frac{d^2\phi^{\mathrm{u}}}{dr^2} + V_{\mathrm{eff}}\phi^{\mathrm{u}} + \left(m^2 - E_{\mathrm{r}}^2\right)\phi^{\mathrm{u}} = 0 \tag{17}$$

where

$$V_{\mathrm{eff}} = \left(eA(r) + \frac{\kappa}{r}\right)^2 - \frac{d}{dr}\left(eA(r) + \frac{\kappa}{r}\right) \tag{18}$$

3. THE COMPLEX DIRAC–ECKART POTENTIAL

The complex Eckart potential which holds discrete energy spectrum is (Znoj:1, 1999)

$$V^{\mathrm{CE}}(x) = \frac{A(A-1)}{\sinh(x)} - 2iB\coth(x) \tag{19}$$

and the corresponding Schrödinger equation is

$$-\frac{d^2}{dx^2} + \frac{A(A-1)}{\sinh(x)} - 2iB\coth(x) - E\psi(x) = 0 \tag{20}$$

Here $A, B \in \Re$. It is obviously seen that the Eckart potential is singular in origin, but one can simply avoid their singularities by a local deformation of the integration path. In [22] by solving the Schrödinger equation (20), the real energy spectrum and corresponding wave functions were found as

$$E_{\mathrm{n}} = \frac{B^2}{(A-n)^2} - (A-n)^2, \quad n = 1, 2, \ldots, n_{\max} < A \tag{21}$$

and

$$\psi_{\mathrm{n}}(x) = \mathcal{N}_{\mathrm{n}}(\coth(x) - 1)^{\mu}(\coth(x) + 1)^{\nu} P_{\mathrm{n}}^{(2\mu,2\nu)}(\coth(x)) \tag{22}$$

where $\mathcal{N}_{\mathrm{n}}$ is a normalization constant, $P_{\mathrm{n}}^{(2\mu,2\nu)}$ is Jacobi polynomial[3] and

$$2\mu = (A-n)$$
$$2\nu = -\frac{iB}{A-n} \tag{23}$$

[3] $P_{\mathrm{n}}^{(2\mu,2\nu)}(x)$ is Jacobi polynomial throughout this article.

Now by defining the complex Dirac–Eckart four vector potential as

$$(eV(r), eA(r)\hat{r}) = \left(i\zeta \coth(r), \left(\frac{C\zeta}{S} \coth(r) - \frac{\kappa}{r} \right) \hat{r} \right), \tag{24}$$

and by using (16), we can obtain

$$V_{\text{eff}} = \frac{\eta(\eta - 1)}{\coth^2(r)} - i\gamma \coth(r) - \eta^2 \tag{25}$$

where $\zeta \in \Re$, $\eta = \frac{\zeta}{S}(S^2 - C^2)$ and $\gamma = 2E_{\text{r}}(C^2 - S^2)\zeta$. So that the second-order differential equation for radial upper component is

$$\left(-\frac{d^2}{dr^2} + \frac{\eta(\eta - 1)}{\coth^2(r)} - i\gamma \coth(r) - \left(E_{\text{r}}^2 + \eta^2 - m^2 \right) \right) \phi_{\text{n}}^{\text{u}}(r) = 0. \tag{26}$$

Comparing (26) with (20) and then using (21) and (22), one can arrive at the relativistic real energy eigenvalue as

$$E_{\text{rn}} = \left[\frac{[m^2 - \eta^2 - (\eta - n)^2](\eta - n)}{[(1 - 2S)\eta - n][(1 + 2S)\eta - n]} \right]^{\frac{1}{2}} \tag{27}$$

in which

$$|\eta_{\max} - \eta| < \sqrt{\eta^2 - m^2} \tag{28}$$

and the correspondence wave function for $\phi_{\text{n}}^{\text{u}}(r)$ is the same eigenfunction in (22) with a new set of parameters as

$$\begin{aligned} 2\mu &= (\eta - n) \\ 2\nu &= -\frac{i\gamma}{\eta - n} = -\frac{i\gamma}{2\mu} \end{aligned} \tag{29}$$

4. THE COMPLEX DIRAC–ROSEN–MORSE II POTENTIAL

The Schrödinger equation for complex Rosen–Morse II potential is

$$-\frac{d^2\psi(x)}{dx^2} + V^{\text{CRM}}(x)\psi(x) - E\psi(x) = 0 \tag{30}$$

where

$$V^{\text{CRM}}(x) = [(b_{\text{R}} + ib_{\text{I}})^2 + q^2 - 1/4] \operatorname{cosch}^2(x) - 2q(b_{\text{r}} + ib_{\text{I}}) \cosh(x) \coth(x) \tag{31}$$

One can rewrite (30) as

$$-\frac{d^2\psi_{\text{n}}(x)}{dx^2} + V^{\text{CRM}}(x)\psi_{\text{n}}(x) = -(q - n - 1/2)^2\psi_{\text{n}}(x) = 0 \tag{32}$$

which shows that

$$E_{\rm n} = -(q - n - 1/2)^2 \tag{33}$$

where

$$n = 0, 1, 2, \ldots, n_{\max} < q - 1/2 \tag{34}$$

and also $q > 1/2$. Hence, we define the Dirac–Rosen–Morse II four vector potential as

$$(eV(r), eA(r)\hat{r}) = \left(0, \left[\zeta \coth(r) - (\eta_{\rm R} + i\eta_{\rm I}) \operatorname{cosch}(r) - \frac{\kappa}{r}\right]\hat{r}\right) \tag{35}$$

Here, $\zeta, \eta_{\rm R}, \eta_{\rm I} \in \mathfrak{R}$. By making use of (18), we find the effective complex Dirac–Rosen–Morse II potential, $V_{\rm eff}^{\rm CDRM}(r)$, as

$$\begin{aligned} V_{\rm eff}^{\rm CDRM}(r) = {} & [(\eta_{\rm R} + i\eta_{\rm I})^2 + (\zeta + 1/2)^2 - 1/4] \operatorname{cosch}^2(r) \\ & -2(\zeta + 1/2)(\eta_{\rm R} + i\eta_{\rm I} \operatorname{cosch}(r) \coth(r) + \zeta^2 \end{aligned} \tag{36}$$

Using (36), we obtain the following second-order differential equation for the upper component, $\phi^{\rm u}(r)$,

$$\begin{aligned} & \left[-\frac{d^2}{dr^2} + [(\eta_{\rm R} + in_{\rm I})^2 + (\zeta + 1/2)^2 - 1/4)] \operatorname{cosch}^2(r) \right. \\ & \left. \quad -2(\zeta + 1/2)(\eta_{\rm R} + i\eta_{\rm I} \operatorname{cosch}(r) \coth(r) \right] \phi_{\rm n}^{\rm u}(r) \\ & = \left(E_{\rm rn}^2 - m^2 - \zeta^2\right) \phi_{\rm n}^{\rm u}(r) \end{aligned} \tag{37}$$

Comparing (37) with Schrödinger equation for complex Rosen–Morse II potential, (31), gives the following real relativistic energy spectrum for complex Dirac–Rosen–Morse II potential as

$$E_{\rm rn} = \sqrt{m^2 + \zeta^2 - (\zeta - n)^2)} \tag{38}$$

where $n = 0, 1, 2, \ldots, n_{\max}$. Here, $\zeta > 0$, and reality of energy spectrum emphasis that $n_{\max}$ satisfy

$$|n_{\max} - \zeta| < \sqrt{m^2 + \zeta^2} \tag{39}$$

5. THE COMPLEX DIRAC–SCARF POTENTIAL

In Znojil (1999), the complex Scarf potential and the corresponding Schrödinger equation is given in the form

$$V^{\rm CS}(x) = [b_{\rm R} + ib_{\rm I})^2 - q^2 + 1/4] \operatorname{sech}^2(x) - 2q(b_{\rm r} + ib_{\rm I}) \operatorname{sech}(x) \tanh(x) \tag{40}$$

$$\ulcorner -\frac{d^2}{dx^2}+\left[\left(b_{\mathrm{R}}+ib_{\mathrm{I}}^2-q^2+1/4\right]\mathrm{sech}^2(x)\right.$$
$$-2q(b_{\mathrm{r}}+ib_{\mathrm{r}})\,\mathrm{sech}(x)\tanh(x)-E^{\urcorner}\psi(x)=0 \tag{41}$$

It is well known that the Schrödinger equation of this potential is exactly solvable. It is easily seen that this complex potential (40) is not invariant under PT-symmetry, but, for the cases which $b_{\mathrm{R}}=0$, is PT-symmetry invariant (where P denotes the parity operator; $(P\psi)(x)=\psi(-x)$ and T the time reversal operator; $(T\psi)(x)=\psi^{*}(x)$). For $q>1/2$, the associated eigenvalues and eigenfunctions are

$$E_{\mathrm{n}}=-(q-n-1/2)^2,\quad n=0,1,2,\ldots,n_{\max}<q-1/2 \tag{42}$$

$$\psi_{\mathrm{n}}(x)=\mathcal{N}_{\mathrm{n}}\,\mathrm{sech}^{q-1/2}(x)e^{\mathrm{ib_I}\arctan(\sinh(x))}P_{\mathrm{n}}^{(-b_{\mathrm{I}}-q,b_{\mathrm{I}}-q)}(i\sinh(x)) \tag{43}$$

Therefore, we define the complex Dirac–Scarf four vector potential as

$$(eV(r),eA(r)\hat{r})=0,(\zeta\tanh(r)-\left(\eta_{\mathrm{R}}+in_{\mathrm{I}}\,\mathrm{sech}(r)-\frac{k}{r}\right)\hat{r} \tag{44}$$

Here, $\zeta,\eta_{\mathrm{R}},\eta_{\mathrm{I}}\in\Re$. By making use of (18), we find the effective complex Dirac–Scarf potential, $V_{\mathrm{eff}}^{\mathrm{CDS}}(r)$, as

$$V_{\mathrm{eff}}^{\mathrm{CDS}}(r)=\ulcorner(\eta_{\mathrm{R}}+i\eta_{\mathrm{I}})^2-(\zeta+1/2)^2+1/4)^{\urcorner}\,\mathrm{sech}^2(r)$$
$$-2(\zeta+1/2)(\eta_{\mathrm{R}}+i\eta_{\mathrm{I}})\,\mathrm{sech}(r)\tanh(r)+\zeta^2 \tag{45}$$

By substituting (45) in (18), we have

$$\ulcorner -\frac{d^2}{dr^2}+[(\eta_{\mathrm{R}}+in_{\mathrm{I}^2})-(\zeta+1/2)^2+1/4)]\,\mathrm{sech}^2(r)$$
$$-2(\zeta+1/2(\eta_{\mathrm{R}}+in_{\mathrm{I}})\,\mathrm{sech}(r)\tan(r)^{\urcorner}\phi_{\mathrm{n}}^{\mathrm{u}}(r)=\left(E_{\mathrm{rn}}^2-m^2-\zeta^2\right)\phi_{\mathrm{n}}^{\mathrm{u}}(r) \tag{46}$$

Comparing (46) with Schrödinger equation for complex Scarf potential, (40), gives the following real relativistic energy spectrum for complex Dirac–Scarf potential as

$$E_{\mathrm{rn}}^2-m^2-\zeta^2=-(\zeta-n)^2 \tag{47}$$

and the corresponding eigenfunction as

$$\psi_{\mathrm{n}}(r)=\mathcal{N}_{\mathrm{n}}\,\mathrm{sech}^{\zeta}(r)e^{i\eta_{\mathrm{I}}\arctan\sinh(\mathrm{r}))}p_{\mathrm{n}}^{-\eta_{\mathrm{I}}-q,\eta_{\mathrm{I}}-q)}(i\sinh(r)) \tag{48}$$

Therefore, for $\zeta>0$ the real relativistic bound state energy eigenvalue is

$$E_{\mathrm{rn}}=\sqrt{m^2+\zeta^2-(\zeta-n)^2)},\quad n=0,1,2,\ldots,n_{\max} \tag{49}$$

where, $n_{\max}$ is the largest positive integer which satisfy

$$|n_{\max}-\zeta|<\sqrt{m^2+\zeta^2} \tag{50}$$

6. THE COMPLEX DIRAC–POSCHL–TELLER POTENTIAL

The complex Poschl–Teller potential is (Znojil, 1999)

$$V^{\text{CPT}}(x) = \frac{M(M-1)}{\sinh^2(t)} - \frac{N(N+1)}{\cosh^2(t)} \tag{51}$$

where $t = x - i\epsilon, x \in (-\infty, +\infty), \epsilon \in (0, \pi/2)$ and $M, N \in \Re$. In this case, $V^{\text{CPT}}(x)$ is not singular at $x = 0$. In [22], the real eigenvalues and corresponding eigenfunctions of the potential (51) were found by solving the Schrödinger equation. It is found

$$E_{\text{n}} = -(2n + \sigma N + \tau M + (\tau - \sigma)/2)^2, \quad n = 1, 2, \ldots, n_{\max} < -\tfrac{1}{2}(\sigma N + \tau M + (\tau - \sigma)/2) \tag{52}$$

where $\tau = \sigma = \pm 1$ and

$$\psi_{\text{n}}(x) = \mathcal{N}_{\text{n}} \sinh^{\tau M}(t) \cosh^{\sigma N+1}(t) P_{\text{n}}^{(\tau M-1/2, \sigma N+1/2)}[\cosh(2t)] \tag{53}$$

So, by defining the complex four vector as

$$(eV(r), eA(r)\hat{r}) = \left(0, \left(\zeta \tanh(t) - \eta \coth(t) - \frac{\kappa}{t}\right)\hat{r}\right) \tag{54}$$

where $t = r - i\epsilon, r \in [0, \infty), \epsilon \in (0, \pi/2)$ and ζ and η are real. We can obtain $V_{\text{eff}}^{\text{CDPT}}$ as

$$V_{\text{eff}}^{\text{CDPT}}(r) = -\zeta(\zeta + 1)\,\text{sech}^2(t) + \eta(\eta + 1)\,\text{cosch}^2(t) + (\zeta - \eta)^2 \tag{55}$$

Therefore, the second-order differential equation for the upper spinor component of Dirac equation for Complex Dirac–Poschl–Teller is

$$\left[-\frac{d^2}{dr^2} + \frac{\eta(\eta+1)}{\sinh^2(\text{t})} - \frac{\zeta(\zeta+1)}{\cosh^2(t)} - \left(E_{\text{rn}}^2 - m^2 - (\zeta - \eta)^2\right]\phi_{\text{n}}^{u}(t) = 0 \tag{56}$$

By comparing (56) with the associated Schrödinger equation of the complex Poschl–Teller potential, (51), we can obtain the relativistic real energy spectrum as

$$E_{\text{rn}} = \sqrt{m^2 - (\zeta - \eta)^2 - (2n + \sigma\zeta + \tau\eta + (\tau - \sigma)/2)^2} \tag{57}$$

where $n = 1, 2, \ldots, n_{\max}$ which, $n_{\max}$ satisfy

$$2n_{\max} < \sqrt{m^2 - (\zeta - \eta)^2} - (\sigma\zeta + \tau\eta + (\tau - \sigma)/2) \tag{58}$$

and also the upper spinor component, $\phi_{\text{n}}^{\text{u}}(r)$, is

$$\phi_{\text{n}}^{\text{u}}(r) = \mathcal{N}_{\text{n}} \sinh^{\tau\eta}(t) \cosh^{\sigma\zeta+1}(t) P_{\text{n}}^{(\tau\eta-1/2, \sigma\zeta+1/2)}[\cosh(2t)] \tag{59}$$

where $\mathcal{N}_{\text{n}}$ is normalization constant and $P_{\text{n}}^{(\mu,\nu)}$ is Jacobi polynomial.

7. CONCLUSION

The non-Hermitian quantum theories have been studied for many complex potentials. It is observed that a relativistic quantum theory need not to be Hermitian to have a real spectrum. In this paper, we obtain the real relativistic energy eigenvalues of a Dirac–charged particle in complex statically and spherically symmetric potentials. We show that these complex Dirac potentials have exact solution for all value of κ(κ is angular momentum quantum number).

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